

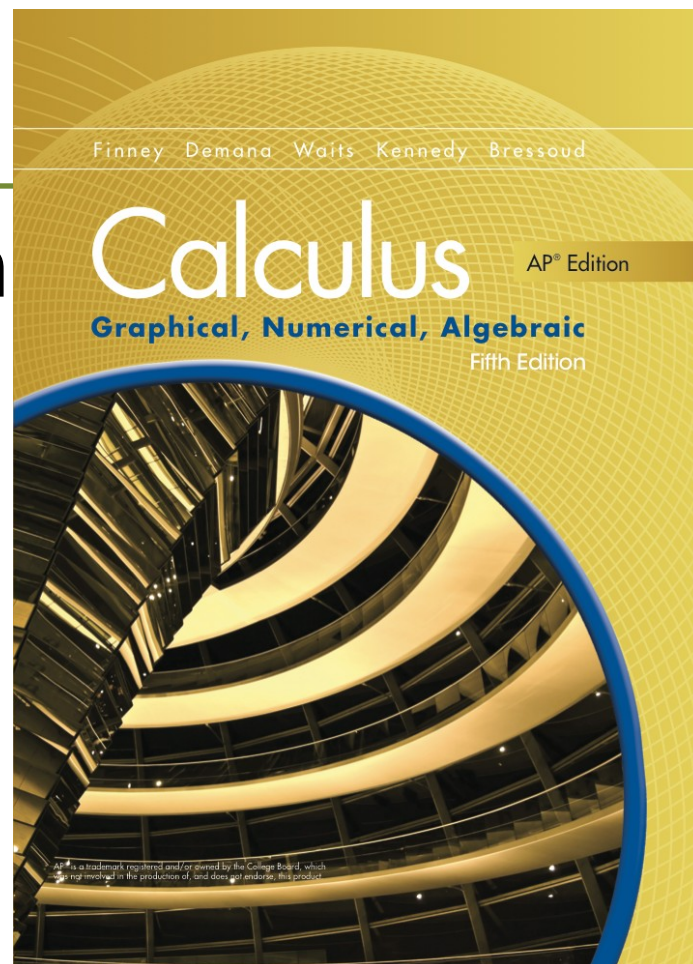
Chapter 3

Derivatives



Section 3.3

Rules for Differentiation



Quick Review

In Exercises 1–6, write the expression as a sum of powers of x .

1. $(x^2 - 2)(x^{-1} + 1)$

2. $\left(\frac{x}{x^2 + 1}\right)^{-1}$

3. $3x^2 - \frac{2}{x} + \frac{5}{x^2}$

4. $\frac{3x^4 - 2x^3 + 4}{2x^2}$

5. $(x^{-1} + 2)(x^{-2} + 1)$

6. $\frac{x^{-1} + x^{-2}}{x^3}$

Quick Review

7. Find the positive roots of the equation $2x^3 - 5x^2 - 2x + 6 = 0$ and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

8. If $f(x) = 7$ for all real numbers x , find

(a) $f(10)$

(b) $f(0)$

(c) $f(x + h)$

(d) $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$

Quick Review

9. Find the derivatives of these functions with respect to x

(a) $f(x) = \pi$

(b) $f(x) = \pi^2$

(c) $f(x) = \pi^{15}$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.

(a) $f(x) = \frac{x}{\pi}$

(b) $f(x) = \frac{\pi}{x}$

Quick Review Solutions

In Exercises 1–6, write the expression as a sum of powers of x

$$1. (x^2 - 2)(x^{-1} + 1)$$

$$x + x^2 - 2x^{-1} - 2$$

$$2. \left(\frac{x}{x^2 + 1} \right)^{-1}$$

$$x + x^{-1}$$

$$3. 3x^2 - \frac{2}{x} + \frac{5}{x^2}$$

$$3x^2 - 2x^{-1} + 5x^{-2}$$

$$4. \frac{3x^4 - 2x^3 + 4}{2x^2}$$

$$\frac{3}{2}x^2 - x + 2x^{-2}$$

$$5. (x^{-1} + 2)(x^{-2} + 1)$$

$$x^{-3} + x^{-1} + 2x^{-2} + 2$$

$$6. \frac{x^{-1} + x^{-2}}{x^3}$$

$$x^{-2} + x^{-5}$$

Quick Review Solutions

7. Find the positive roots of the equation $2x^3 - 5x^2 - 2x + 6 = 0$ and evaluate the function $y = 500x^6$ at each root. Round your answers to the nearest integer, but only in the final step.

At $x = 1.173$, $500x^6 \approx 1305$

At $x = 2.394$, $500x^6 \approx 94,212$

8. If $f(x) = 7$ for all real numbers x , find

(a) $f(10)$ 7

(b) $f(0)$ 7

(c) $f(x + h)$ 7

(d) $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ 0

Quick Review Solutions

9. Find the derivatives of these functions with respect to x

(a) $f(x) = \pi$

$$0$$

(b) $f(x) = \pi^2$

$$0$$

(c) $f(x) = \pi^{15}$

$$0$$

10. Find the derivatives of these functions with respect to x using the definition of the derivative.

(a) $f(x) = \frac{x}{\pi}$

$$f'(x) = \frac{1}{\pi}$$

(b) $f(x) = \frac{\pi}{x}$

$$f'(x) = -\pi x^{-2}$$

What you'll learn about

- Power functions
- Sum and difference rules
- Product and quotient rules
- Negative integer powers of x
- Second and higher order derivatives

... and why

These rules help us find derivatives of functions analytically in a more efficient way.

Rule 1 Derivative of a Constant Function

If f is the function with the constant value c , then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0$$

This means that the derivative of every constant function is the zero function.

Rule 2 Power Rule for Positive Integer Powers of x .

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

The Power Rule says:

To differentiate x^n , multiply by n and subtract 1 from the exponent.

Rule 3 The Constant Multiple Rule

If u is a differentiable function of x and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}$$

This says that if a differentiable function is multiplied by a constant, then its derivative is multiplied by the same constant.

Rule 4 The Sum and Difference Rule

If u and v are differentiable functions of x , then their sum and differences are differentiable at every point where u and v are differentiable.

At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

Example Positive Integer Powers, Multiples, Sums, and Differences

Differentiate the polynomial $y = x^4 + 2x^2 - \frac{3}{4}x - 19$

That is, find $\frac{dy}{dx}$.

By Rule 4 we can differentiate the polynomial term-by-term, applying Rules 1 through 3.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(2x^2) - \frac{d}{dx}\left(\frac{3}{4}x\right) - \frac{d}{dx}(19) \quad \text{Sum and Difference Rule}$$

$$= 4x^3 + 2 \cdot 2x - \frac{3}{4} - 0 \quad \text{Constant and Power Rules}$$

$$= 4x^3 + 4x - \frac{3}{4}$$

Example Positive Integer Powers, Multiples, Sums, and Differences

Does the curve $y=x^4 - 8x^2 + 2$ have any horizontal tangents?

If so, where do they occur?

Verify your result by graphing the function.

If any horizontal tangents exist, they will occur where the slope $\frac{dy}{dx}$ is equal to zero. To find these points we will set $\frac{dy}{dx}=0$ and solve for x

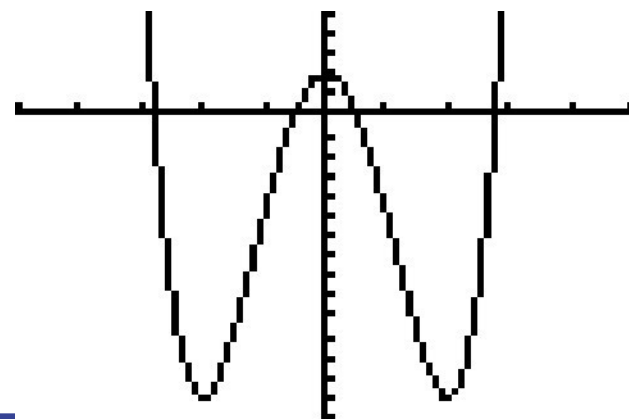
Calculate $\frac{dy}{dx} = \frac{d}{dx}(x^4 - 8x^2 + 2) = 4x^3 - 16x$

Set $\frac{dy}{dx}=0$ and solve for x

$$4x^3 - 16x = 0$$

$$4x(x^2 - 4) = 0; \quad 4x = 0 \quad x^2 - 4 = 0$$

This gives horizontal tangents at $x=0, 2, -2$.



Rule 5 The Product Rule

The product of two differentiable functions u and v is differentiable, and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

The derivative of a product is actually the sum of two products.

Example Using the Product Rule

Find $f'(x)$ if $f(x) = (x^3 - 4)(x^2 + 3)$

Using the Product Rule with $u = x^3 - 4$ and $v = x^2 + 3$, gives

$$\begin{aligned} f'(x) &= \frac{d}{dx} [(x^3 - 4)(x^2 + 3)] = (x^3 - 4)2x + (x^2 + 3)3x^2 \\ &= 2x^4 - 8x + 3x^4 + 9x^2 \\ &= 5x^4 + 9x^2 - 8x \end{aligned}$$

Rule 6 The Quotient Rule

At a point where $v \neq 0$, the quotient $y = \frac{u}{v}$ of two differentiable functions is differentiable, and

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Since order is important in subtraction, be sure to set up the numerator of the Quotient rule correctly.

Example Using the Quotient Rule

Find $f'(x)$ if $f(x) = \frac{x^3 - 4}{x^2 + 3}$

Using the Quotient Rule with $u = x^3 - 4$ and $v = x^2 + 3$, gives

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left[\frac{(x^3 - 4)}{(x^2 + 3)} \right] = \frac{(x^2 + 3) 3x^2 - (x^3 - 4) 2x}{(x^2 + 3)^2} \\ &= \frac{3x^4 + 9x^2 - 2x^4 + 8x}{(x^2 + 3)^2} \\ &= \frac{x^4 + 9x^2 + 8x}{(x^2 + 3)^2} \end{aligned}$$

Rule 7 Power Rule for Negative Integer Powers of x

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

This is basically the same as Rule 2 except now n is negative.

Example Negative Integer Powers of x

Find an equation for the line tangent to the curve $y = \frac{1}{x}$ at the point $(1, 1)$.

Rewrite the function as $y = x^{-1}$ and use the Power Rule to find the derivative.

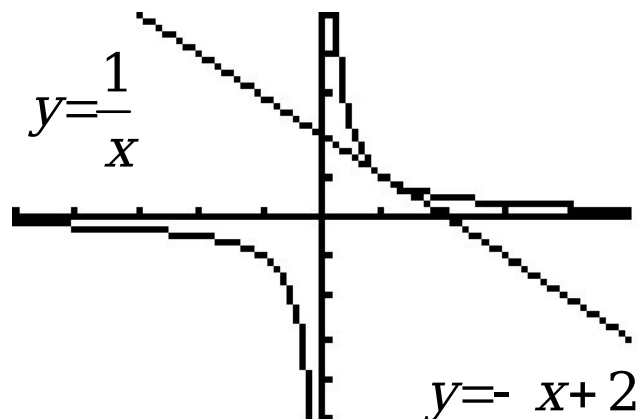
$$y' = -1x^{-2} = -\frac{1}{x^2}$$

Evaluate $y'(1) = -\frac{1}{1} = -1$

The line through $(1, 1)$ with slope $m = -1$ is

$$y - 1 = -1(x - 1)$$

$$y = -x + 2$$



This shows the graph of the function and its tangent line at $(1, 1)$.

Second and Higher Order Derivatives

The derivative $y' = \frac{dy}{dx}$ is called the *first derivative* of y with respect to x .

The first derivative may itself be a differentiable function of x . If so,

its derivative, $y'' = \frac{dy'}{dx} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2},$

is called the *second derivative* of y with respect to x . If y'' (y double prime) is differentiable, its derivative,

$$y''' = \frac{dy''}{dx} = \frac{d^3 y}{dx^3},$$

is called the *third derivative* of y with respect to x .

Second and Higher Order Derivatives

The multiple-prime notation begins to lose its usefulness after three primes.

So we use $y^{(n)} = \frac{d}{dx} y^{(n-1)}$ “ y super n ”

to denote the n th derivative of y with respect to x

Do not confuse the notation $y^{(n)}$ with the n th power of y , which is y^n .

Quick Quiz Sections 3.1 – 3.3

You may use a graphing calculator to solve the following problems.

1. Let $f(x) = |x + 1|$. Which of the following statements about f are true?

- I. f is continuous at $x = -1$.
- II. f is differentiable at $x = -1$.
- III. f has a corner at $x = -1$.

- (A) I only
- (B) II only
- (C) III only
- (D) I and III only

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Quick Quiz Sections 3.1 – 3.3

2. If the line normal to the graph of f at the point $(1, 2)$ passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1)$ =?

(A) -2

(B) 2

(C) $-\frac{1}{2}$

(D) $\frac{1}{2}$

(E) 3

Quick Quiz Sections 3.1 – 3.3

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(A) -2

(B) 2

(C) $-\frac{1}{2}$

(D) $\frac{1}{2}$

(E) 3

Quick Quiz Sections 3.1 – 3.3

3. Find $\frac{dy}{dx}$ if $y = \frac{4x-3}{2x+1}$.

(A) $\frac{10}{(4x-3)^2}$

(B) $-\frac{10}{(4x-3)^2}$

(C) $\frac{10}{(2x+1)^2}$

(D) $-\frac{10}{(2x+1)^2}$

(E) 2

Quick Quiz Sections 3.1 – 3.3

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(D) $-\frac{10}{(2x+1)^2}$

(E) 2